

# Supersymmetric M5 Brane Theories on $R \times CP^2$

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## Abstract

We propose 4 and 12 supersymmetric Yang-Mills-Chern-Simons theories on  $R \times CP^2$  obtained by twisted  $Z_k$  moddings and dimensional reduction of the 6d (2,0) superconformal field theories on  $R \times S^5$ . These theories have a discrete coupling constant  $\frac{1}{g_{YM}^2} = \frac{k}{4\pi^2}$  so that instantons represent the Kaluza-Klein modes correctly. We calculate the perturbative part of the  $SU(N)$  gauge group Euclidean path integral for the index function and confirm it with the known half-BPS index. The scalar and fermionic fields have the conformal dimension prescribed by the 6d theory. From the similar twisted  $Z_k$  modding of the  $AdS_7 \times S^4$  geometry, we speculate that the  $M$  region is for  $k \lesssim N^{1/3}$  and the type IIA region is  $N^{1/3} \lesssim k \lesssim N$ . When nonperturbative corrections are included, our theory is expected to produce the full index of the 6d (2,0) theory.

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# 1 Introduction

The physics of M5 branes [1, 2] remains as one of great mysteries in M-theory [1, 3]. Some fundamental structures of the underlying 6d (2,0) theory with superconformal  $OSp(2, 6|2)$  symmetry are not yet known. One of a promising approach is the 5d maximally super symmetric gauge theories whose instantons may provide all Kaluza-Klein physics of the circle compactified 6d theory [4, 5]. The study of a BPS sector by index calculation seems to provide the exact results on the DLCQ limit of the 6d (2,0) theory [6, 7]. However, one wants to have more tools to probe this 6d theory, for example, to calculate the full index function on  $S^1 \times S^5$ .

In this work we propose exactly such a tool. First we put the 6d (2,0) theory on  $R \times S^5$ . The five sphere is a circle fibration over  $CP^2$ , and we mod out the theory by  $Z_k$  along this fiber direction with some additional twisting along a  $Sp(2)_R = SO(5)$  symmetry direction. This allows a consistent truncation of the 6d theory to a 5d theory on  $R \times CP^2$  with some supersymmetries. While we do not know the exact nonabelian 6d (2,0) theory, one can find the consistent 5d theory which has both Yang-Mills term and Chern-Simons term of type  $JAdA$  with Kähler form  $J$  on  $CP^2$ . This  $Z_k$  modding and dimensional reduction lead to the overall coupling constant  $1/g_{YM}^2 = k/4\pi^2$  and the 5d theory has a weakly coupled regime for  $k > N$ . The 5d theory and its amount of supersymmetry depends on the twisting. Here we construct the 5d theories with 4 and 12 supersymmetries.

As instantons in the 5d maximally supersymmetric Yang-Mills theory provides the Kaluza-Klein modes [8, 9], instantons solitons on  $CP^2$  in our theories are expected to provide also the KK physics. Our theory has the weak coupling limit and may be complete by its own once the non-perturbative physics is included for all  $k = 1, 2, 3 \dots$ .

While our theories does not have the standard conformal symmetry on 5d, they have still some superconformal symmetries and allow the definition of the superconformal indices. Here we calculate the conformal index in the large  $k$  or free theory limit and found that it matches exactly what is expected.

The AdS geometry can be obtained by a similar  $Z_k$  modding of  $AdS_7 \times S^5$  [10]. This geometry is unusual as the asymptotic geometry is not  $AdS_6$ . We speculate that there are three regions of  $k$ : the M-theory region for  $1 \leq k \lesssim N^{1/3}$ , the type IIA region for  $N^{1/3} \lesssim K \lesssim N$ , and the high curvature region for  $N \lesssim k$ .

Our approach is inspired in part by the ABJM theory on M2 branes which has  $Z_k$  modding of the  $SO(8)$  R-symmetry [11]. Our  $Z_k$  modding is on both the space  $S^5$  and the part of the scalar field space  $R^2$ . The number of supersymmetries can be reduced to 4 or 12 depending on the twisting on the field space  $R^2$  and should get enhanced to 32 for  $k = 1$ . Also there is no fixed point in the AdS geometry and so there is no twisted sector. Of course there are many differences between the ABJM theory and our theory as our theory is for M5 branes not M2 branes and also the space  $CP^2$  is compact instead of non-compact.

We have noticed that 5d  $JFA$  type Chern-Simons term has appeared in Ref. [12, 13] while their setting is different from ours. There is another work by one of us (HK) and Seok Kim where the index on M5 brane has been approached by the 5d Yang-Mills theory on  $S^5$  [14]. Not only perturbative calculation on  $S^5$  is done explicitly but also a conjecture on instanton part has been provided. This is another approach to the index calculation of the 6d (2,0) theory. More relevant for the future work would be the index calculation on  $S^1 \times S^4$  done recently for the 5d superconformal field theories [15]. There are some related recent works [16–18] on the supersymmetric theories on  $S^5$ .

Our theories on  $R \times CP^2$  has the Hamiltonian whose eigenvalues are the conformal dimension of the states on  $CP^2$ . We will argue that the superconformal symmetries for 4 and 12 supersymmetric theories are  $SU(1|2)$  and  $SU(3|2)$ , respectively. Our theory does not appear in the standard classification of the super conformal field theories as there is no Poincare supersymmetry in our theory [19, 20].

This 5d theory, obtained after the  $Z_k$  modding and the dimensional reduction, has the weak coupling limit and so could be ultraviolet complete when nonperturbative part is included. Of course our theories is not defined on  $R^{1+4}$  and so the usual perturbative expansion is not available. All the fields belong to the adjoint representation of the gauge group and the overall coupling constant is given as  $1/g_{YM}^2 = k/4\pi^2$ . There is only one length scale, that is, the radius  $r$  of the  $S^5$ . The large  $k$  limit is the weak coupling limit and  $k = 1$  is the strong coupling limit. For the  $SU(N)$  or  $U(N)$  theory, there is also 'tHooft coupling constant  $\lambda = N/k$ . For large 'tHooft coupling limit has the natural AdS limit. The geometry obtained by the  $Z_k$  modding of the  $Ad_7 \times S^4$  is rather complicated as the boundary geometry is a  $Z_k$  modding of the boundary geometry  $S^5 \times S^4$ .

The index function for a conformal field theory is an important tool to explore the theory [21–23]. The index function of the 6d (2,0) theory on  $S^1 \times S^5$  is one of the major interest. The index for the  $U(1)$  theory on a single M5 has been done [24]. Our 5d theories have both perturbative parts and instanton parts. In this work, we restrict ourself to just the perturbative part and found it matches with the known 1/2 BPS index on the single M5 brane [25].

The outline of this work is as follows. In Sec. 2 we start with the 6d abelian (2,0) theory and do the twisted  $Z_k$  modding and the dimensional reduction to obtain some supersymmetric 5d Yang-Mills Chern-Simons theories on  $R \times CP^2$ . In Sec. 3 we explore the properties of these theories, including the spectrum of the abelian theory. In Sec.4 we introduce the index function and calculate it by the Euclidean path integral in the weak coupling limit. In Sec.5 we conclude with some remarks. In Appendices we include the properties of  $CP^2$ , the gamma matrix and spinor convention and the Killing spinors on  $S^5$  and  $CP^2$ .

## 2 5d Supersymmetric Lagrangian on $R \times CP^2$

Let us start with the 6d abelian (2,0) theory on  $R^{1+5}$  for the field  $B_{MN}, \lambda, \phi_I (I = 1, 2, 3, 4, 5)$ . The 3-form field strength  $H = dB$  should be selfdual  $H = *H$ . We start with the supersymmetric action with additional spectator field  $H = -*H$  which does not get involved in the supersymmetric transformation [26]. The bosonic part of the superconformal symmetry  $OSp(2,6|2)$  is made of the  $SO(2,6)$  conformal symmetry and  $Sp(2) = SO(5)$  R-symmetry. The conformal dimension of  $H, \lambda, \phi_I$  is  $3, 5/2, 2$ , respectively. One does the radial quantization and obtain the action in  $R \times S^5$ . The Cartan of the spatial rotation group  $SU(4) = SO(6)$  is  $j_1, j_2, j_3$  and the Cartan of the R-symmetric group  $Sp(2)_R = SO(5)$  is  $R_1, R_2$ . The  $R_1$  rotates the scalar fields  $\phi_1, \phi_2$  and  $R_2$  rotates  $\phi_4, \phi_5$ . Spinor field  $\lambda$  belongs to  $\mathbf{4}$  of  $SU(4)$  and  $\mathbf{4}$  of  $Sp(2)_R$ . Both the fermion field  $\lambda$  and supercharge  $Q$  transform identically under  $SU(4)$  and  $Sp(2)_R$ . In terms of roots  $\pm e_i \pm e_j, (i, j = 1, 2, 3)$  of  $SO(6)$ , the spinor representations  $\mathbf{4}$  and  $\bar{\mathbf{4}}$  are given by the weights  $(\pm e_1 \pm e_2 \pm e_3)/2$  with odd or even numbers of minus signs.

Let us do the radial quantization of the (2,0) theory on  $R \times S^5$ . See the appendix A for the metrics on  $S^5$  and  $CP^2$ . The action on  $R \times S^5$  is

$$S = \int_{R \times S^5} dt d\Omega_{S^5} \left\{ -\frac{1}{12} H_{MNP} H^{MNP} - \frac{i}{2} \bar{\lambda} \Gamma^M \hat{\nabla}_M \lambda - \frac{1}{2} \partial_M \phi_I \partial^M \phi_I - \frac{2}{r^2} \phi_I \phi_I \right\}. \quad (2.1)$$

The supersymmetric transformation for the tensor multiplet is

$$\begin{aligned} \delta B_{MN} &= -\bar{\lambda} \Gamma_{MN} \epsilon = -\bar{\epsilon} \Gamma_{MN} \lambda, \\ \delta \phi_I &= -\bar{\lambda} \rho_I \epsilon = +\bar{\epsilon} \rho_I \lambda, \\ \delta \lambda &= +\frac{i}{6} H_{MNP} \Gamma^{MNP} \epsilon + i \partial_M \phi_I \Gamma^M \rho_I \epsilon + \frac{2i}{3} \phi_I \rho_I \Gamma^M \hat{\nabla}_M \epsilon, \\ \delta \bar{\lambda} &= -\frac{i}{6} H_{MNP} \bar{\epsilon} \Gamma^{MNP} + i \partial_M \phi_I \bar{\epsilon} \Gamma^M \rho_I + \frac{2i}{3} \hat{\nabla}_M \bar{\epsilon} \Gamma^M \rho_I \phi_I. \end{aligned} \quad (2.2)$$

The Killing spinors  $\epsilon$  should satisfy

$$\hat{\nabla}_M \epsilon = \frac{i}{2R} \Gamma_M \tilde{\epsilon}, \quad \Gamma^M \hat{\nabla}_M \tilde{\epsilon} = 2i\epsilon, \quad (2.3)$$

which can be solved by  $\tilde{\epsilon} = \pm \Gamma_0 \epsilon$ .

Note that

$$H_{MNP} \Gamma^{MNP} \epsilon = \frac{1}{2} (H_{MNP} + *H_{MNP}) \Gamma^{MNP} \epsilon, \quad (2.4)$$

where

$$*H_{MNP} = \frac{1}{6} \epsilon_{MNPQRS} H^{QRS}, \quad \epsilon_{0123456} = -1. \quad (2.5)$$

Only the selfdual part  $H = {}^*H$  appears in the supersymmetry transformation. Thus the anti-selfdual part of the field strength transform as

$$\delta(H_{MNP} - {}^*H_{MNP}) = i\bar{\epsilon}\Gamma_{MBP}\Gamma^Q\partial_Q\lambda, \quad (2.6)$$

which vanishes on-shell.

We can introduce the radius of the five sphere anytime to fix the dimension and so we drop them except when it is more convenient. The metric for the five sphere is

$$ds_{S^5}^2 = ds_{\mathbb{CP}^2}^2 + (dy + V)^2, \quad (2.7)$$

where  $0 \leq y \leq 2\pi$ . The Kähler form  $J$  is given by

$$J = \frac{1}{2}dV. \quad (2.8)$$

We want to a  $Z_k$  modding along the fiber direction

$$y \sim y + \frac{2\pi}{k} \quad (2.9)$$

on the 6d (2,0) theory.

The Killing spinors as shown in appendix B have nontrivial  $y$ -dependence and such  $Z_k$  modding would remove them unless one introduce a twisting along some direction of  $Sp(2) = SO(5)$  R-symmetry. Let us consider the plus sign case with first  $\tilde{\epsilon} = +\Gamma_0\epsilon$ . With the notation for the eigenspinors  $\gamma^{12}\epsilon^{s_1s_2} = i s_1\epsilon^{s_1s_2}$ ,  $\gamma^{34}\epsilon^{s_1s_2} = i s_2\epsilon^{s_1s_2}$ , we group the 16 Killings  $\epsilon_+$  to the 4 and 12 spinors. The first group of the Killing spinors is made of

$$(\mathbf{I}) \quad \epsilon_+ \sim e^{-\frac{i}{2}t + \frac{3i}{2}y} \epsilon_0^{++}, \quad (2.10)$$

with constant spinors  $\epsilon_0^{++}$  which form a singlet of  $SU(3)$  isometry of  $\mathbb{CP}^2$  and the **4** fundamental representation of  $Sp(2)_R = SO(5)$ . The second group of the three independent Killing spinors such that

$$(\mathbf{II}) \quad \epsilon_+ \sim e^{-\frac{i}{2}t - \frac{i}{2}y} (\epsilon_1^{+-}, \epsilon_2^{-+}, \epsilon_3^{--}), \quad (2.11)$$

where the eigenspinors are a complicated matrix linear combinations of three constant spinors. They form a triplet of  $SU(3)$  isometry of  $\mathbb{CP}^2$  and the **4** fundamental representation of  $Sp(2)_R = SO(5)$ . The exact form is not important here.

We want to cancel the  $y$ -dependence of the spinor parameter by introducing a twisting of the spinor parameter along the  $R$ -symmetry direction. There are many equivalent choices and also less supersymmetric choices. Here we choose two choices for the simplicity and also we twist both spinor and scalar fields also to be consistent with the supersymmetric transformation.

The first choice is to introduce new variables

$$(\mathbf{I}) \quad \epsilon_{old} = e^{-\frac{3\rho_{45}}{2}y} \epsilon_{new}, \quad \lambda_{old} = e^{-\frac{3\rho_{45}}{2}y} \lambda_{new}, \quad (\phi_4 + i\phi_5)_{old} = e^{+3iy} (\phi_4 + i\phi_5)_{new}. \quad (2.12)$$

This change of variables leads to

$$(I) \quad \partial_y \rightarrow \partial_y + 3iR_2 \quad (2.13)$$

on new variables. Here  $R_2$  is one of the Cartan of  $Sp(2)_R = SO(5)$  R-symmetry. The corresponding  $U(1)_{R_2}$  transformation is  $\phi_4 + i\phi_5 \rightarrow e^{i\alpha}(\phi_4 + i\phi_5)$  and  $\lambda \rightarrow e^{-\frac{\rho_{45}}{2}\alpha}\lambda$ . The new spinor parameter has the  $y$ -dependence as  $e^{3(i+\rho_{45})y/2}\epsilon_0^{++}$ , and so we choose the constant spinor to satisfy  $\rho_{45}\epsilon_0 = -i$  to remove the  $y$ -dependence. This supersymmetry would survive the  $Z_k$  modding. This would restrict the possible  $R$ -charge of allowed  $\epsilon_+$  spinors to be complex two depending on the eigenvalue  $\rho_{12}$ . As  $\epsilon_-$  is a complex conjugate, there would be four surviving supersymmetries in the first case after  $Z_k$  modding.

The second one is to introduce new variables so that

$$(II) \quad \epsilon_{old} = e^{+\frac{\rho_{45}}{2}y}\epsilon_{new}, \quad \lambda_{old} = e^{+\frac{\rho_{45}}{2}y}\lambda_{new}, \quad (\phi_4 + i\phi_5)_{old} = e^{-iy}(\phi_4 + i\phi_5)_{new}. \quad (2.14)$$

This leads to the change of the derivative  $\partial_y$  on new fields to

$$(II) \quad \partial_y \rightarrow \partial_y - iR_2. \quad (2.15)$$

The  $y$ -dependence of the Killing spinors  $\epsilon_+$  can be removed once  $\rho_{45}\epsilon_+ = -i\epsilon_+$ . These three Killing spinors would survive the  $Z_k$  modding and so the resulting theory would have 12 supersymmetries.

The  $Z_k$  modding of the new spinor and scalar fields to be

$$\begin{aligned} (I) \quad & \lambda(y)_{old} \sim e^{-\frac{3\pi\rho_{45}}{k}}\lambda(y + \frac{2\pi}{k})_{old}, \quad (\phi_4 + i\phi_5)(y)_{old} \sim e^{+\frac{6\pi i}{k}}(\phi_4 + i\phi_5)(y + \frac{2\pi}{k})_{old}, \\ (II) \quad & \lambda(y)_{old} \sim e^{+\frac{\pi\rho_{45}}{k}}\lambda(y + \frac{2\pi}{k})_{old}, \quad (\phi_4 + i\phi_5)(y)_{old} \sim e^{-\frac{2\pi i}{k}}(\phi_4 + i\phi_5)(y + \frac{2\pi}{k})_{old}. \end{aligned} \quad (2.16)$$

Such consistent  $Z_k$  modding of the 6d (2,0) theory reduces the number of super symmetries. Still we do not know the exact form of the resulting 6d theory.

Let us now do the dimensional reduction of the theory to 5d by requiring the new variables to be independent of  $y$ . Then the  $y$ -independent new spinor and scalar fields are invariant under the  $Z_k$  modding and so are allowed. As we found the  $y$  independent Killing spinors for the superconformal transformation, the supersymmetric transformation under these Killing spinors will not introduce additional  $y$  dependence between the fields. This  $Z_k$  modding would shrink the circle fiber size relative to other scale and so the theory would become more close to the 5d theory.

Now one can keep only  $y$ -independent modes and write down the 5d theory on  $R \times CP^2$ . This theory would have still nontrivial number of super symmetries and superconformal super symmetries. There would be nonperturbative effects including instantons in this 5d theory and they would provide all Kaluza-Klein mode physics of the 6d (2,0) theory as what is expected for 5d maximally Yang-Mills theory on  $R^{1+4}$ .

We first consider only the action for the scalar and spinor fields and then fix the gauge kinetic term to complete the supersymmetry for the abelian case. Then one generalize the theory to nonabelian case. Only difficulty is to choice the right normaliaztion for the coupling constant. For both cases we argue in the next section that the 5d gauge coupling constant is given by

$$\frac{1}{g_{YM}^2} = \frac{k}{4\pi^2 r}, \quad (2.17)$$

where  $r$  is the radius of the  $S^5$  sphere and is regard as unity as it is only length scale of the theory. The theory becomes weakly coupled in large  $k$  limit. As the fields are in the adjoint representation of the gauge group, one expect the presence of 't Hooft coupling constant

$$\frac{N}{k} \quad (2.18)$$

for  $U(N)$  gauge theory.

To be more explicit we use the four-component 5d notation for the spinors as given in appendix A. The reality condition becomes

$$\lambda = BC\lambda^*, \quad \epsilon = BC\epsilon^*. \quad (2.19)$$

The Killing spinor equation for the  $y$ -independent new spinor parameter  $\epsilon_{new}$  becomes

$$\partial_t \epsilon = \frac{i}{2} \gamma_0 \tilde{\epsilon}, \quad D_m \epsilon = -\frac{i}{2} J_{mn} \gamma^n \epsilon + \frac{i}{2} \gamma_m \tilde{\epsilon}, \quad (2.20)$$

where  $m = 1, 2, 3, 4$ . The spinor variables satisfy additional conditions for two cases:

$$\begin{aligned} \text{(I)} \quad & \rho_{45} \epsilon_+ = -i \epsilon_+, \quad D_a = \nabla_a + \frac{3\rho_{45}}{2} V_a, \quad \tilde{\epsilon} = -\left[3\rho_{45} + \frac{1}{2} J_{ab} \gamma^{ab}\right] \epsilon, \\ \text{(II)} \quad & \rho_{45} \epsilon_+ = -i \epsilon_+, \quad D_a = \nabla_a - \frac{\rho_{45}}{2} V_a, \quad \tilde{\epsilon} = \left[\rho_{45} - \frac{1}{2} J_{ab} \gamma^{ab}\right] \epsilon, \end{aligned} \quad (2.21)$$

where  $\nabla_a$  is the spinor covariant derivative on  $\text{CP}^2$ . The  $U(1)$  rotation by  $R_2$  are deformed to a  $U(1)'_R$  due to the twisting for both cases.

The resulting 4 supersymmetric nonabelian 5d action on  $\text{R} \times \text{CP}^2$  for the first case is

$$\begin{aligned} S_{\text{I}} = \frac{k}{4\pi^2} \int_{\text{R} \times \text{CP}^2} d^5 x \quad & \text{tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma\eta} J_{\mu\nu} \left( A_\rho \partial_\sigma A_\eta - \frac{i}{3} A_\rho A_\sigma A_\eta \right) \right. \\ & - \frac{1}{2} D_\mu \phi_I D^\mu \phi_I + \frac{1}{4} [\phi_I, \phi_J]^2 + \frac{i}{3} \epsilon_{abc} \phi_a [\phi_b, \phi_c] - 2\phi_a^2 - \frac{13}{2} \phi_i^2 \\ & \left. - \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda - \frac{i}{2} \bar{\lambda} \rho_I [\phi_I, \lambda] - \frac{1}{8} \bar{\lambda} \gamma^{mn} \lambda J_{mn} + \frac{3}{4} \bar{\lambda} \rho_{45} \lambda \right], \end{aligned} \quad (2.22)$$

where  $I = 1, 2, 3, 4, 5$ ,  $a = 1, 2, 3$ ,  $i = 4, 5$  and

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \\ D_\mu \phi_a &= \partial_\mu \phi_a - i[A_\mu, \phi_a], \\ D_\mu \phi_i &= \partial_\mu \phi_i - i[A_\mu, \phi_i] + 3V_\mu \epsilon_{ij} \phi_j, \\ D_\mu \lambda &= \left[ \partial_\mu \lambda + \frac{1}{4} \omega_\mu^{ab} \gamma^{ab} + \frac{3}{2} V_\mu \rho_{45} \right] \lambda - i[A_\mu, \lambda]. \end{aligned} \quad (2.23)$$

Of course only spatial components of  $J_{\mu\nu}$  are non-zero. The supersymmetric transformation

$$\begin{aligned} \delta A_\mu &= +i\bar{\lambda}\gamma_\mu \epsilon = -i\bar{\epsilon}\gamma_\mu \lambda, \\ \delta \phi_I &= -\bar{\lambda}\rho_I \epsilon = \bar{\epsilon}\rho_I \lambda, \\ \delta \lambda &= +\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon + iD_\mu \phi_I \rho_I \gamma^\mu \epsilon - \frac{i}{2} [\phi_I, \phi_J] \rho_{IJ} \epsilon - 3\epsilon_{ij} \phi_i \rho_j \epsilon - 2\phi_I \rho_I \tilde{\epsilon}. \end{aligned} \quad (2.24)$$

The supercharge  $Q$  is a singlet under  $SU(3)$  isometry of  $\mathbb{CP}^2$  and a doublet under  $SU(2)_R$  with nontrivial  $U(1)'_R$  charge. Thus the supergroup behind the first model would be  $SU(1|2)$ .

The 12 supersymmetric 5d action on  $R \times \mathbb{CP}^2$  for the second case is

$$\begin{aligned} S_{\text{II}} &= \frac{k}{4\pi^2} \int_{R \times \mathbb{CP}^2} d^5x \, \text{tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma\eta} J_{\mu\nu} \left( A_\rho \partial_\sigma A_\eta - \frac{i}{3} A_\rho A_\sigma A_\eta \right) \right. \\ &\quad - \frac{1}{2} D_\mu \phi_I D^\mu \phi_I + \frac{1}{4} [\phi_I, \phi_J]^2 + \frac{i}{3} \epsilon_{abc} \phi_a [\phi_b, \phi_c] - 2\phi_a^2 - \frac{5}{2} \phi_i^2 \\ &\quad \left. - \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda - \frac{i}{2} \bar{\lambda} \rho_I [\phi_I, \lambda] - \frac{1}{8} \bar{\lambda} \gamma^{mn} \lambda J_{mn} - \frac{1}{4} \bar{\lambda} \rho_{45} \lambda \right], \end{aligned} \quad (2.25)$$

where  $I = 1, 2, 3, 4, 5$ ,  $a = 1, 2, 3$ ,  $i = 4, 5$  and

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \\ D_\mu \phi_a &= \partial_\mu \phi_a - i[A_\mu, \phi_a], \\ D_\mu \phi_i &= \partial_\mu \phi_i - i[A_\mu, \phi_i] - V_\mu \epsilon_{ij} \phi_j, \\ D_\mu \lambda &= \left[ \partial_\mu \lambda + \frac{1}{4} \omega_\mu^{ab} \gamma^{ab} - \frac{1}{2} V_\mu \rho_{45} \right] \lambda - i[A_\mu, \lambda]. \end{aligned} \quad (2.26)$$

The supersymmetric transformation is

$$\begin{aligned} \delta A_\mu &= i\bar{\lambda}\gamma_\mu \epsilon = -i\bar{\epsilon}\gamma_\mu \lambda, \\ \delta \phi_I &= -\bar{\lambda}\rho_I \epsilon = \bar{\epsilon}\rho_I \lambda, \\ \delta \lambda &= +\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon + iD_\mu \phi_I \rho_I \gamma^\mu \epsilon - \frac{i}{2} [\phi_I, \phi_J] \rho_{IJ} \epsilon + \epsilon_{ij} \phi_i \rho_j \epsilon - 2\phi_I \rho_I \tilde{\epsilon}. \end{aligned} \quad (2.27)$$

The supercharge  $Q$  is a triplet under  $SU(3)$  isometry of  $\mathbb{CP}^2$  and a doublet under  $SU(2)_R$  with nontrivial  $U(1)'_R$  charge. Thus the supergroup behind the second model would be  $SU(3|2)$ .



### 3 Properties of the 5d theories

There are several properties of these 5d theories we want to study in this work. While we do not explore in the present work, the instantons in our theories would play the Kaluza-Klein modes for the circle fiber as the case in the maximally supersymmetric 5d Yang-Mills theory on  $R^5$ .

The instanton number on  $CP^2$  is

$$\nu = \frac{1}{8\pi^2} \int_{CP^2} \text{Tr}(F \wedge F). \quad (3.1)$$

The Hamiltonian of the 6d (2,0) theory on  $R \times S^5$  has the conformal dimension as the eigenvalues. Similarly the Hamiltonian for our 5d theories would have the conformal dimension as the eigenvalues. The abelian scalar field harmonics on  $S^5$  is summarised in detail in the appendix which shows that the lowest conformal dimension for the untwisted scalar field  $\phi_a$  is two as expected. Upon  $Z_k$  modding, first nontrivial Kaluza-Klein modes start with conformal dimension  $k + 2$ . Such KK modes along the circle fiber is supposed to be represented by instantons in the 5d theory. As a single instanton has mass  $4\pi^2/g_{YM}^2$  with our normalisation  $1/(4g_{YM}^2)\text{Tr}F^2$  where  $F$  is  $N \times N$  hermitian matrix valued two-form for  $U(N)$  gauge group and the KK modes has mass  $k$ , the inverse coupling coefficient  $1/g_{YM}^2$  is chosen to be  $k/4\pi^2$ . The instantons on  $CP^2$  has been explored before in Ref. [27]. It would be interesting to consider their work in our index calculation context.

The strongest coupling occurs at  $k = 1$ . For this value, the supersymmetry should be enhanced to maximal value and the global symmetry should be enhanced to  $OSp(8|2)$  supergroup. The instantons should play the essential role here.

One could ask whether instantons and anti-instantons are BPS in our 5d theories. Here we just consider the selfdual or anti-selfdual gauge field strength, leaving the study of the instanton solution itself to the next paper. For the first case (I), we note that  $\gamma_{1234}\epsilon = -\epsilon$  and only anti-instantons can be BPS. For the second case (II), both instantons and anti-instantons can be BPS. The amount of preserved super symmetries is interesting also. For the first case the anti-instantons preserve all of 4 susy. For the second case the anti-instantons preserve 4 susy and instantons preserve 8 susy.

The 6d expectation is that the conformal dimension of the scalar, spinor and gauge field of our 5 theories would be 2, 5/2, 3 as expected for their 6d origin. For the scalar and spinor fields, the  $S^5$  harmonic analysis shows that it is the case indeed. For the vector potential, we shows that is case in Sec.

While the instanton mass fixes the coupling constant, its 6d field theoretic origin can

be read as follows:

$$\begin{aligned}
S_{6d} &= \int_{\mathbb{R} \times S^5} d^6x \left( -\frac{1}{2} \partial_\mu (\phi_I)_{6d} \partial^\mu (\phi_I)_{6d} + \dots \right) \\
\rightarrow S_{5d} &= \frac{2\pi r}{k} \int_{\mathbb{R} \times CP^2} d^5x \left( -\frac{1}{2} \partial_\mu (\phi_I)_{6d} \partial^\mu (\phi_I)_{6d} + \dots \right) \\
\rightarrow S_{5d} &= \frac{k}{4\pi^2 r} \int_{\mathbb{R} \times CP^2} d^5x \left( -\frac{1}{2} \partial_\mu (\phi_I)_{5d} \partial^\mu (\phi_I)_{5d} + \dots \right)
\end{aligned} \tag{3.2}$$

with the mass dimension for the 6d field  $\phi_I$  being two and the mass dimension for the final 5d field  $\phi_I$  being one. The scalar field is rescaled so that

$$(\phi_I)_{5d} = \frac{2\pi\sqrt{2\pi}r}{k} (\phi_I)_{6d} \tag{3.3}$$

Once the abelian theory is obtained, one can complete the nonabelian generalization.

The quadratic Chern-Simons term has been noted before [12,13]. A beautiful argument in Ref. [12] is that the  $y$  independent field equation for the 3-form tensor field on  $R \times S^5$  leads naturally to the presence of the quadratic Chern-Simons term. Another argument is that the instantons are KK modes along the fiber direction and KK gauge field  $V_p dx^p$  is a gauge field on  $CP^2$  space with magnetic field  $2J$ . Whenever the instanton moves, it feels the background magnetic field and so the interaction term should be proportional to  $V_p dx^p/dt$  where  $x^p$  is the position of a point like instanton on  $CP^2$ . The natural field theoretic expression is then the Chern-Simons term. We do not know any argument yet similar to the quantization of 3d Chern-Simons level. It would be interesting to find more argument to support our choice of the coupling constant. The full effect of this Chern-Simons term is not clear at this moment.

While there is a Meyer's term in the potential, the vacuum structure of the 5d theories does not have any degenerate vacua. The scalar potential has the minimum only at the symmetric point. The study of the gaugino transformation shows also that the vacuum is unique.

The Gauss law in the  $U(1)$  theory implies

$$\frac{k}{4\pi^2} D_m F^{m0} + \frac{k}{4\pi^2} e^{0mnpq} J_{mn} F_{pq} = 0. \tag{3.4}$$

Total charge should be zero in the compact  $CP^2$ . As  $J$  is selfdual, the anti-selfdual flux

$$F \sim e^1 \wedge e^2 - e^3 \wedge e^4 \tag{3.5}$$

seems possible without violation of the Gauss law but it does not satisfy  $dF = 0$ . The selfdual configuration  $F = 2J$  in the abelian theory has the instanton number  $1/2$  which is not allowed due to the Gauss law. In nonabelian theories, there could be nontrivial vector boson charge and so the Gauss law could be satisfied nontrivially. There may be some monopole-like operator as in 3d case [23].

While the second case has more super symmetries, the first case is simpler as the Killing spinor is constant spinor on  $\mathbb{CP}^2$ . We will focus on the first case from now on. Still, it is hard to penetrate the detail physics of the theory yet. We do not see any restriction on the gauge group unlike the 6d theory [1].

Let us briefly mention the spectrum of the theory for first type (I) for abelian case. The detail spectrum for the scalar and fermion fields on  $S^5$  is given in Ref. [14]. The spectrum for the vector field on  $\mathbb{CP}^2$  will be given in next section. As the index calculation in the next section provides the detail of the spectrum on  $\mathbb{CP}^2$ , here we just focus on the scalar field spectrum. The spectrum of a scalar field of conformal dimension 2 on  $R \times S^5$  has the mass

$$(-\nabla_{S^5}^2 + 4)Y^{\ell_1, \ell_2} = (\ell_1 + \ell_2 + 2)^2 Y^{\ell_1, \ell_2}, \quad -i\partial_y Y^{\ell_1, \ell_2} = (\ell_1 - \ell_2)Y^{\ell_1, \ell_2}. \quad (3.6)$$

The highest weight of the given irreducible representation  $Y^{\ell_1, \ell_2}$  would be  $\ell_1 w_1 + \ell_2 w_2$  with two fundamental weights  $w_1, w_2$  of  $SU(3)$ . The dimension of the representation of the highest weight  $(\ell_1, \ell_2)$  is  $(\ell_1 + 1)(\ell_2 + 1)(\ell_1 + \ell_2 + 2)/2$ . For  $\phi_{1,2,3}$  fields, there is no twisting.  $Z_k$  modding puts the constraints  $\ell_1 - \ell_2 = kn$  with integer  $n$ . The  $y$ -independent mode with  $Y^{\ell, \ell}$  has the spectrum on  $CP^2$  as

$$(-\nabla_{\mathbb{CP}^2}^2 + 4)Y^{\ell, \ell} = 4(\ell + 1)^2 Y^{\ell, \ell}, \quad (3.7)$$

with the degeneracy  $2(\ell + 1)^3$ . Note that the conformal dimension of this mode is  $\varepsilon = 2\ell + 2$  and so it starts from 2 as we expect for the scalar field in the 6d theory. The first KK mode would with either  $(\ell_1, \ell_2) = (k, 0)$  or  $(0, k)$ . Both of them has the conformal dimension  $\varepsilon = k + 2$  with degeneracy  $(k + 1)(k + 2)/2$ .

The twisted mode  $\phi_4 + i\phi_5$  has more complicated  $y$ -independent modes and KK modes. The  $y$  independent mode is given by  $Y^{\ell, \ell+3}$  or  $Y^{\ell+3, \ell}$  with conformal dimension  $\varepsilon = 2\ell + 5$ . One can do the similar analysis for the fermion field whose conformal dimension on  $CP^2$  starts from  $\varepsilon = 5/2$  as expected for the 6d fermion. The vector field analysis done in next section shows that its conformal dimension on  $\mathbb{CP}^2$  starts from  $\varepsilon = 4$  not 3 which is expected for the 3 form tensor field in 6d. There may be no constant three form among the harmonics on  $S^5$ . Instantons with perturbative effects should reproduce the KK modes. We hope to come back to these issues near future.

## 4 Superconformal index

We will now define the superconformal index on 6d (2,0) theory and analyze its properties upon the  $Z_k$  modding introduced in section 2. Later we will relate this index with the 5d index on  $R \times \mathbb{CP}^2$  where the full 6d index should be obtainable from 5d computation involving the nonperturbative effect. The superconformal index encodes the spectrum of the BPS states of the radially quantized theory on  $R \times S^5$ . More precisely, the index we

will define shortly counts the BPS states annihilated by a chosen supercharges  $Q$  and its conjugate  $S$  among 4 supercharges. The chosen supercharge  $Q$  satisfies the algebra

$$\{Q, S\} = \varepsilon - j_1 - j_2 - j_3 + 2R_1 + 2R_2 \equiv \Delta, \quad (4.1)$$

and hence we count the BPS states saturating the bound  $\Delta = 0$ . Here the supercharge  $Q$  has charges under the symmetry rotation as  $j_1 = j_2 = j_3 = -\frac{1}{2}$ ,  $R_1 = R_2 = -\frac{1}{2}$ .

The superconformal index of the (2,0) theory is defined as

$$I(x, y_1, y_2, q) = \text{tr} \left[ (-1)^F x^{\varepsilon+R_1} y_1^{j_1-j_2} y_2^{j_2-j_3} q^j \right], \quad (4.2)$$

where  $x = e^{-\beta}$ ,  $y_1 = e^{-i\gamma_1}$ ,  $y_2 = e^{-i\gamma_2}$  denote the chemical potentials for the Cartan generators of the symmetry commuting with  $Q$ , and  $j = j_1 + j_2 + j_3 - 3R_2$ . This index for a single M5-brane and the gravity dual theory at large  $N$  is studied in [?]. As the abelian (2,0) theory is free, one can easily obtain the index for the single M5-brane theory by reading off the BPS letters from the field content of the (2,0) theory. The index of the  $U(1)$  (2,0) theory is given by the Plethystic exponential of the single letter index  $f$

$$I = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f(x^n, y_i^n, q^n) \right],$$

$$f(x, y_1, y_2, q) = \frac{x + x^2 q^3 - x^2 q^2 (1/y_1 + y_1/y_2 + y_2) + x^3 q^3}{(1 - xqy_1)(1 - xqy_2/y_1)(1 - xq/y_2)}. \quad (4.3)$$

The denominator comes from the derivatives, the first two terms of the numerator comes from the scalar fields, three minus term in the numerator comes from the spinor fields, and the last term in the numerator comes from the spinor field equation. There is no contribution from the three form selfdual tensor field. One interesting limit of this index is to take  $q \rightarrow 0$  limit where the index reduces to the half-BPS index that is the index function of the half-BPS states (preserving 16 supersymmetries). In this limit, the letter index simply becomes  $f = x$  and it reflects that only a single complex scalar  $\phi_1 - i\phi_2$  contributes to the index. The  $A_{N-1} = SU(N)$  non-abelian version of the half-BPS index [?] is already given by

$$I_{1/2\text{-BPS}} = \prod_{m=1}^N \frac{1}{1 - x^m}. \quad (4.4)$$

This is the index we will reproduce in this section by calculating the perturbative part of the corresponding Euclidean path integral on  $S^1 \times \mathbb{CP}^2$ .

Now we turn to the  $Z_k$  modding of the superconformal index. We introduced in section 2 the  $Z_k$  quotient along the circular fiber direction  $y$  twisted by  $R_2$  rotation. The  $j$  corresponds to the rotation of this twisted  $y$  direction. The modding leaves only the  $Z_k$  singlet states carrying  $j = kn$  ( $n \in \mathbb{Z}$ ) charge and truncates all other states. Accordingly, the index of the 6d theory with  $Z_k$  quotient is defined as

$$I_{Z_k} = \text{tr} \left[ (-1)^F x^{\varepsilon+R_1} y_1^{j_1-j_2} y_2^{j_2-j_3} q^j \right] \Big|_{j=kn}. \quad (4.5)$$

When  $k = 1$ , it reproduces the index for the (2,0) theory discussed above. On the other hand, at infinite  $k$  limit or the free coupling limit, all the KK states with non-zero  $j$  charge are truncated and the index reduces to the 5d index counting the BPS states of the free theory limit on  $R \times CP^2$ . This limit is achieved by taking  $q \rightarrow 0$  limit in the superconformal index. Here, we note that this index at infinite  $k$  coincides with the half-BPS index (4.4) as two limits imply identically  $q \rightarrow 0$ .

We expect that the 5d index including the non-perturbative states can reproduce the full 6d superconformal index. The 5d theory of the first case (I) introduced in section 2 preserves the same supercharge  $Q$  used to define the 6d index, and, therefore, we can define the 5d index in the same way as the 6d index (4.2). The perturbative states in 5d theory correspond to the  $j$  singlet modes while the instanton states represent the KK states with non-zero  $j$  charge. We then identify the instanton number with the KK momentum number  $j$ .

The index can be considered as the path integral of the Euclidean action of the 5d theory on  $S^1 \times CP^2$

$$I(x, y_i, q) = \int_{S^1 \times CP^2} \mathcal{D}\Psi e^{-S_I^E[\Psi]}. \quad (4.6)$$

The twisted boundary condition along the time circle  $S^1$  of radius  $\beta r$  is considered. The Euclidean version of the action (2.22) is given by

$$\begin{aligned} S_I^E = & \frac{k}{4\pi^2 r} \int_{S^1 \times CP^2} d^5x \operatorname{tr} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \epsilon^{\mu\nu\lambda\rho\sigma} J_{\mu\nu} \left( A_\lambda \partial_\rho A_\sigma - \frac{2i}{3} A_\lambda A_\rho A_\sigma \right) \right. \\ & + \frac{1}{2} D_\mu \phi_I D^\mu \phi_I - \frac{1}{4} [\phi_I, \phi_J]^2 + \frac{i}{3r} \epsilon^{abc} [\phi_a, \phi_b] \phi_c + \frac{2}{r^2} (\phi_a)^2 + \frac{13}{2r^2} (\phi_i)^2 \\ & \left. - \frac{i}{2} \lambda^\dagger \gamma^\mu D_\mu \lambda - \frac{i}{2} \lambda^\dagger \rho_I [\lambda, \phi_I] - \frac{1}{8r} \lambda^\dagger J_{\mu\nu} \gamma^{\mu\nu} \lambda + \frac{3}{4r} \lambda^\dagger \rho_{45} \lambda \right], \end{aligned} \quad (4.7)$$

where the fermion  $\lambda$  is subject to the reality condition  $\lambda = BC\lambda^*$  and the radius  $r$  of  $S^5$  is introduced. The twisted boundary condition shifts the time derivative such as

$$\partial_\tau \rightarrow \partial_\tau + \frac{\beta}{\beta r} R_1 + \frac{i\gamma_1}{\beta r} (j_1 - j_2) + \frac{i\gamma_2}{\beta r} (j_2 - j_3), \quad (4.8)$$

and, from now on, we consider the time derivatives as this shifted one. The action is invariant under the supersymmetry transformation

$$\begin{aligned} \delta \phi_I &= -\lambda^\dagger \rho_I \epsilon, \\ \delta A_\mu &= -i\lambda^\dagger \gamma_\mu \epsilon, \\ \delta \lambda &= \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon - iD_\mu \phi_I \gamma^\mu \rho_I \epsilon - \frac{i}{2} [\phi_I, \phi_J] \rho_{IJ} \epsilon + \frac{3}{r} \epsilon_{ij} \phi_i \rho_j \epsilon - \frac{2i}{r} \phi_I \rho_I \tilde{\epsilon}. \end{aligned} \quad (4.9)$$

The supersymmetry parameter  $\epsilon$  satisfy the conditions

$$D_\mu \epsilon = -\frac{i}{2r} J_{\mu\nu} \gamma^\nu \epsilon + \frac{1}{2r} \gamma_\mu \tilde{\epsilon}, \quad \frac{3}{2} \rho^{45} \epsilon = -\frac{1}{4} J_{\mu\nu} \gamma^{\mu\nu} \epsilon + \frac{i}{2} \tilde{\epsilon}, \quad \tilde{\epsilon} = i\rho^{45} \gamma_\tau \epsilon, \quad (4.10)$$

and we found four solutions to these conditions,

$$\gamma_{12}\epsilon_+ = \gamma_{45}\epsilon_+ = -\rho^{45}\epsilon_+ = i\epsilon_+, \quad (4.11)$$

and its conjugation  $\epsilon_- = BC\epsilon_+^*$ . It turns out that the four Killing spinors are convariantly constant on  $\mathbb{CP}^2$

$$D_m\epsilon_\pm = 0 \quad (m = 1, 2, 3, 4). \quad (4.12)$$

We would like to evaluate the superconformal index using the localization technique. The localization would lead to the path integral over the instanton configuration on  $\mathbb{CP}^2$  base. The calculation of the nonperturbative instanton contributions will be a future work.

At infinite  $k$ , the gaussian integral of the quadratic equations produces the exact result. For convenience, let us divide the field content to a vector multiplet and an adjoint hypermultiplet (though there is no notion of the hypermultiplet as the theory preserves only 4 supercharges). We first pick up a complex supercharge  $Q$  corresponding to  $\rho_{12}\epsilon = -i\epsilon$  and decompose the spinors as

$$\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.13)$$

Then the vector multiplet consists of  $A_\mu, \chi, \phi_3$  and the hypetmultiplet consists of two complex scalar  $q^A$  and a complex fermion  $\psi$  defined as

$$q_1 \equiv \frac{1}{\sqrt{2}}(\phi_4 - i\phi_5), \quad q_2 \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \psi \equiv \psi^2. \quad (4.14)$$

The action with the new fields becomes

$$\begin{aligned} S_{\mathbf{I}}^E = & \frac{k}{4\pi^2} \int_{\mathbb{R} \times \mathbb{CP}^2} d^5x \, \text{tr} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \epsilon^{\mu\nu\lambda\rho\sigma} J_{\mu\nu} \left( A_\lambda \partial_\rho A_\sigma - \frac{2i}{3} A_\lambda A_\rho A_\sigma \right) \right. \\ & + \frac{1}{2} D_\mu \phi_3 D^\mu \phi_3 + |D_\mu q^A|^2 + \frac{2}{r^2} (\phi_3)^2 + \frac{4}{r^2} |q^2|^2 + \frac{13}{r^2} |q^1|^2 \\ & + |[\phi_3, q^A]|^2 + \frac{1}{2} |[q^A, \bar{q}_A]|^2 + \frac{1}{2} (\sigma^I)^A{}_B (\sigma^I)^C{}_D [q^B, \bar{q}_A] [q^D, \bar{q}_C] - \frac{2}{r} \phi_3 [q^2, \bar{q}_2] \\ & - \frac{i}{2} \chi^\dagger \gamma^\mu D_\mu \chi - i\psi \gamma^\mu D_\mu \psi - \frac{1}{8r} \chi^\dagger J_{\mu\nu} \gamma^{\mu\nu} \chi - \frac{1}{4r} \psi^\dagger J_{\mu\nu} \gamma^{\mu\nu} \psi + \frac{3i}{4r} \chi^\dagger \sigma^3 \chi + \frac{3i}{2r} \psi^\dagger \psi \\ & \left. - \frac{i}{2} \chi^\dagger [\phi_3, \chi] + i\psi^\dagger [\phi_3, \psi] + \sqrt{2}i\psi^\dagger [\chi_A, q^A] - \sqrt{2}i[\bar{q}_A, \chi^\dagger] \psi \right], \quad (4.15) \end{aligned}$$

where  $\sigma^{I=1,2,3}$  are the Pauli matrices.

Before performing the path integral, let us first fix the gauge following [21]. We choose the Coulomb gauge  $D^m A_m = 0$  and impose the residual gauge fixing condition as  $\frac{d}{d\tau} \alpha = 0$  where  $\alpha \equiv \frac{1}{\omega_{CP^2}} \int_{CP^2} A_\tau$  is the s-wave component (or holonomy) of  $A_\tau$ . The holonomy  $\alpha$  is the only zero mode of the quadratic action. The residual gauge fixing introduces the Haar

measure to the path integral. Thus the index at large  $k$  becomes the integral of the 1-loop determinant by the holonomy  $\alpha$

$$I = \frac{1}{N!} \int \prod_{i=1}^N \left[ \frac{d\alpha_i}{2\pi} \right] \prod_{i<j}^N \left[ 2 \sin \left( \frac{\alpha_i - \alpha_j}{2} \right) \right]^2 \times I_{1-loop}. \quad (4.16)$$

To obtain the 1-loop determinant, we will use the various  $CP^2$  harmonics carrying electric charges  $R_2$ . Some of them are constructed in [14, 30]. Let us first focus on the scalars in the hypermultiplet. The scalars have the following quadratic terms

$$\bar{q}_1 \left[ -D_\tau^2 - D^m D_m + \frac{13}{r^2} \right] q^1 + \bar{q}_2 \left[ -D_\tau^2 - D^m D_m + \frac{4}{r^2} \right] q^2, \quad (4.17)$$

where the time derivative is

$$D_\tau = \partial_\tau - i[\alpha, \ ] + \frac{\beta}{\beta r} R_1 + \frac{i\gamma_1}{\beta r} (j_1 - j_2) + \frac{i\gamma_2}{\beta r} (j_2 - j_3). \quad (4.18)$$

We need to use the charged  $SU(3)$  harmonics  $Y^{l+3R_2, l}$  if  $R_2 > 0$  or  $Y^{l, l+3|R_2|}$  if  $R_2 < 0$  according to  $R_2$  charges of the scalar fields. Here, the charged harmonics  $Y^{l_1, l_2}$  carries  $R_2$  charge  $\frac{l_1 - l_2}{3}$ . Then the corresponding harmonics are  $Y^{l, l+3}$  for  $q^1$  and  $Y^{l, l}$  for  $q^2$  respectively, and they diagonalize the quadratic equation. The 1-loop determinant of the hyper scalars becomes

$$\begin{aligned} \det_{H,b} = & \prod_{\alpha \in \text{root}} \prod_{l=0} \prod_{m_1, m_2 \in (l, l+3)} \sin \left( \frac{\alpha - m_i \gamma_i - i(2l+5)\beta}{2} \right) \sin \left( \frac{\alpha - m_i \gamma_i + i(2l+5)\beta}{2} \right) \\ & \times \prod_{\alpha \in \text{root}} \prod_{l=0} \prod_{m_1, m_2 \in (l, l)} \sin \left( \frac{\alpha - m_i \gamma_i - i(2l+1)\beta}{2} \right) \sin \left( \frac{\alpha - m_i \gamma_i + i(2l+3)\beta}{2} \right). \end{aligned} \quad (4.19)$$

where  $m_i \gamma_i = m_1 \gamma_1 + m_2 \gamma_2$  and  $m_i$  denote the two Cartan charges of  $(l_1, l_2)$  representation for  $SU(3)$  isometry.

For the complex fermion  $\psi$ , we introduce the four spinor basis on  $CP^2$

$$\Psi_1 = Y^{l, l+3} \epsilon_+, \quad \Psi_2 = \gamma^\tau \gamma^m D_m Y^{l, l+3} \epsilon_+, \quad \Psi_3 = Y^{l, l} \epsilon_-, \quad \Psi_4 = \gamma^\tau \gamma^m D_m Y^{l, l} \epsilon_-, \quad (4.20)$$

where  $Y^{l_1, l_2}$  is the charged  $SU(3)$  harmonics defined above. These four basis can diagonalize the fermion quadratic action

$$\psi^\dagger \left[ -i\gamma^\tau D_\tau - iD^m \gamma_m - \frac{1}{4r} J_{mn} \gamma^{mn} + \frac{3i}{2r} \right] \psi. \quad (4.21)$$

One then obtain the 1-loop determinant for the fermion field in the hypermultiplet

$$\begin{aligned} \det_{H,f} = & \prod_{\alpha \in \text{root}} \prod_{l=0} \prod_{m_1, m_2 \in (l, l+3)} \sin \left( \frac{\alpha - m_i \gamma_i - i(2l+5)\beta}{2} \right) \sin \left( \frac{\alpha - m_i \gamma_i + i(2l+5)\beta}{2} \right) \\ & \times \sin \left( \frac{\alpha + 3i\beta}{2} \right) \prod_{\alpha \in \text{root}} \prod_{l=1} \prod_{m_1, m_2 \in (l, l)} \sin \left( \frac{\alpha - m_i \gamma_i - i(2l+1)\beta}{2} \right) \sin \left( \frac{\alpha - m_i \gamma_i + i(2l+3)\beta}{2} \right). \end{aligned} \quad (4.22)$$

The first line corresponds to the 1-loop determinant from  $\Psi_1, \Psi_2$  and the second line is from  $\Psi_3, \Psi_4$ . Combining the complex scalar and the fermion contributions, the final 1-loop determinant of the hypermultiplet is given by

$$\frac{\det_{H,f}}{\det_{H,b}} = \prod_{\alpha \in \text{root}} \frac{1}{\sin\left(\frac{\alpha - i\beta}{2}\right)} \sim \exp \left[ \sum_{n=1}^{\infty} \sum_{i,j} \frac{1}{n} x^n e^{ni\alpha_{ij}} \right]. \quad (4.23)$$

Let us move on to the vectormultiplet contribution. It is straightforward to compute the fermionic contribution by using the same spinor basis (4.20). The quadratic equation for  $\chi^1$  is given by

$$(\chi^1)^\dagger \left[ -i\gamma^\tau D_\tau - i\gamma^m D_m - \frac{1}{4r} J_{mn} \gamma^{mn} + \frac{3i}{2r} \right] \chi^1. \quad (4.24)$$

The corresponding 1-loop determinant becomes

$$\begin{aligned} \det_{V,f} = & \prod_{\alpha \in \text{root}} \prod_{l=0} \prod_{m_1, m_2 \in (l, l+3)} \sin\left(\frac{\alpha - m_i \gamma_i - i(2l+6)\beta}{2}\right) \sin\left(\frac{\alpha - m_i \gamma_i + i(2l+4)\beta}{2}\right) \\ & \times \sin\left(\frac{\alpha + 2i\beta}{2}\right) \prod_{\alpha \in \text{root}} \prod_{l=1} \prod_{m_1, m_2 \in (l, l)} \sin\left(\frac{\alpha - m_i \gamma_i - i(2l+2)\beta}{2}\right) \sin\left(\frac{\alpha - m_i \gamma_i + i(2l+2)\beta}{2}\right). \end{aligned} \quad (4.25)$$

The first line is again the 1-loop determinant of  $\Psi_1, \Psi_2$  and the second line is from  $\Psi_3, \Psi_4$ .

The quadratic action of the vector field is

$$\begin{aligned} & \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + i\epsilon^{\mu\nu\lambda\rho\sigma} A_\mu \partial_\nu A_\lambda J_{\rho\sigma} = (D_m A_\tau)^2 + 2A_\tau \partial_\tau D_m A^m \\ & - A_m (D_\tau^2 \delta_n^m + D^2 \delta_n^m - D^m D_n - 6) A^n + 4i A_\tau D_m A_n J^{mn} - 2i A_m D_\tau A_n J^{mn}. \end{aligned} \quad (4.26)$$

We find that the following vector harmonics form the complete basis of the 5 vector components

$$\mathcal{A}_\tau = Y^{l,l}, \quad \mathcal{A}_m^1 = D_m Y^{l,l}, \quad \mathcal{A}_m^2 = J_{mn} D^n Y^{l,l}, \quad \mathcal{A}_m^3 = \epsilon_-^\dagger \gamma_m \gamma^n D_n Y^{l,l+3} \epsilon_+. \quad (4.27)$$

Here,  $\mathcal{A}_\tau, \mathcal{A}^1, \mathcal{A}^2$  are real vectors and  $\mathcal{A}^3$  is a complex vector. As we have already taken into account the zero mode of  $A_\tau$ , which gives the holonomy  $\alpha$  and Haar measure of the gauge group, the range of the harmonics  $Y^{l,l}$  is therefore  $l > 0$ . Under the Coulomb gauge  $D^m A_m = 0$ , we can turn off the modes corresponding to  $\mathcal{A}_m^1$ . The other two real vectors  $\mathcal{A}_\tau, \mathcal{A}_m^2$  mix each other in the quadratic action. Taking into account the determinant factors from the gauge fixing procedure, we obtain the 1-loop determinant for the real vectors

$$\prod_{\alpha \in \text{root}} \prod_{l=1} \prod_{m_1, m_2 \in (l, l)} \left[ \sin\left(\frac{\alpha - m_i \gamma_i - i(2l+2)\beta}{2}\right) \sin\left(\frac{\alpha - m_i \gamma_i + i(2l+2)\beta}{2}\right) \right]^{\frac{1}{2}}. \quad (4.28)$$



The complex vector  $\mathcal{A}^3$  is an eigenvector of the quadratic equation (4.26) and its 1-loop determinant is

$$\prod_{\alpha \in \text{root}} \prod_{l=0} \prod_{m_1, m_2 \in (l, l+3)} \sin \left( \frac{\alpha - m_i \gamma_i - i(2l+6)\beta}{2} \right) \sin \left( \frac{\alpha - m_i \gamma_i + i(2l+4)\beta}{2} \right). \quad (4.29)$$

We then collect the fermion and the vector contributions as well as the contribution from  $\phi^3$ . After the huge cancellation between the fermionic and bosonic contributions, we finally find that the 1-loop determinant of the vector multiplet is trivial

$$\frac{\det_{V,f}}{\det_{V,b}} = 1. \quad (4.30)$$

Combining the contributions from the vector and the hypermultiplet, we obtain the following superconformal index at infinite  $k$

$$\begin{aligned} I(x, y_1, y_2)_{k \rightarrow \infty} &= \frac{1}{N!} \int \prod_{i=1}^N \left[ \frac{d\alpha_i}{2\pi} \right] \prod_{i < j}^N \left[ 2 \sin \left( \frac{\alpha_i - \alpha_j}{2} \right) \right]^2 \exp \left[ \sum_{n=1}^{\infty} \sum_{i,j} \frac{1}{n} x^n e^{ni\alpha_{ij}} \right] \\ &= \prod_{m=1}^N \frac{1}{1 - x^m}. \end{aligned} \quad (4.31)$$

It follows that the index receives the contributions from the states formed by a single letter  $\phi_1 + i\phi_2$ . This result agrees with the 6d superconformal index at infinite  $k$  and, therefore, agrees with the half-BPS index (4.4). We believe that the full superconformal index at finite  $k$  can be calculated by including the instanton contribution.

## 5 Supergravity

Let us briefly consider the  $\text{AdS}_7 \times \text{S}^4$  geometry corresponding to the 6d (2,0) theory [10]. In case we need the complete  $\text{AdS}_7$  geometry with  $S^5$  boundary. The maximally supersymmetric  $\text{AdS}_7 \times \text{S}^4$  geometry is

$$\begin{aligned} ds^2 &= R^2 (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_5^2) + \frac{1}{4} R^2 d\Omega_4^2, \\ F_4 &\sim N\epsilon_4, \quad R/\ell_p = 2(\pi N)^{1/3}. \end{aligned} \quad (5.1)$$

The 5d unit sphere and 4d unit sphere are modded by

$$\frac{\text{S}^5 \times \text{S}^4}{Z_k}. \quad (5.2)$$

The metrics on  $S^5$  and  $S^4$  are, respectively,

$$\begin{aligned} ds_{S^5}^2 &= ds_{CP^2}^2 + (dy' + V)^2, \\ ds_{S^4}^2 &= d\vartheta^2 + \sin^2 \vartheta d\chi'^2 + \cos^2 \vartheta ds_{S^2}. \end{aligned} \quad (5.3)$$

where  $\chi'$  is the phase corresponding to the phase of  $\phi_4 + i\phi_5$  and  $dV = 2J$  is the Kähler 2-form on  $\mathbb{CP}^2$ . The  $Z_k$  modding for the first and second cases are

$$\begin{aligned} \text{(I)} \quad (y', \chi') &\sim (y', \chi') + \frac{2\pi}{k}(1, 3), \\ \text{(II)} \quad (y', \chi') &\sim (y', \chi') + \frac{2\pi}{k}(1, -1). \end{aligned} \quad (5.4)$$

Let us focus on the first case with the change of coordinates to

$$y' = \frac{y}{k}, \quad \chi' = \chi + \frac{3y}{k}, \quad (5.5)$$

with  $y \in [0, 2\pi]$  and  $\chi \in [0, 2\pi]$ . The geometry becomes

$$\begin{aligned} ds^2 &= R^2 \left[ -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho ds_{\mathbb{CP}^2}^2 + \frac{1}{k^2} \sinh^2 \rho (dy + kV)^2 \right] \\ &\quad + \frac{R^2}{4} \left[ d\vartheta^2 + \sin^2 \vartheta (d\chi + \frac{3dy}{k})^2 + \cos^2 \vartheta ds_{S^2}^2 \right], \\ F &\sim N(\mathcal{V}_{S^4} + \frac{3}{k} \sin \vartheta \cos^2 \vartheta \, d\vartheta \wedge dy \wedge \mathcal{V}_{S^2}), \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} ds_{S^4}^2 &= d\vartheta^2 + \sin^2 \vartheta dy^2 + \cos^2 \vartheta ds_{S^2}^2, \\ \mathcal{V}_{S^4} &= \sin \vartheta \cos^2 \vartheta \, d\vartheta \wedge d\chi \wedge \mathcal{V}_{S^2}. \end{aligned} \quad (5.7)$$

where  $\mathcal{V}_{S^2}$  is the volume form of a unit 2-sphere.

The corresponding Type IIA geometry can be obtained by the relation:

$$\begin{aligned} ds_{11}^2 &= e^{-2\sigma/3} ds_{10}^2 + e^{4\sigma/3} (dy + \mathcal{A})^2, \\ F_{11}^4 &= e^{4\sigma/3} F_{10}^4 + e^{\sigma/3} F_{10}^3 \wedge dy. \end{aligned} \quad (5.8)$$

Some of NS-NS fields of  $\sigma, g_{MN}, B_{MN}$  and R-R fields  $C_\mu, C_{\mu\nu\rho}$  are nonvanishing as  $C_M dx^M = \mathcal{A}$  and  $dB = -\frac{1}{k} \sin \vartheta \cos^2 \vartheta d\vartheta \wedge \mathcal{V}_{S^2}$ . The metric (5.6) containing  $(dy + kV)^2$  and  $(d\chi + 3dy/k)^2$  becomes

$$\frac{R^2}{4k^2} (4 \sinh^2 \rho + 9 \sin^2 \vartheta) (dy + \mathcal{A})^2 + \frac{R^2 \sinh^2 \rho \sin^2 \vartheta}{4 \sinh^2 \rho + 9 \sin^2 \vartheta} (d\chi - 3V)^2, \quad (5.9)$$

where

$$\mathcal{A} = k \frac{4 \sinh^2 \rho V + 3 \sin^2 \vartheta d\chi}{4 \sinh^2 \rho + 9 \sin^2 \vartheta}. \quad (5.10)$$

Thus the relation (5.8) implies

$$e^{4\sigma/3} = \frac{R^2}{4k^2} (4 \sinh^2 \rho + 9 \sin^2 \vartheta), \quad (5.11)$$

and

$$e^{-2\sigma/3} ds_{10}^2 = + R^2 [-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho ds_{CP^2}^2] + \frac{R^2}{4} (d\vartheta^2 + \cos^2 \vartheta ds_{S^2}^2) + \frac{R^2 \sin^2 \rho \sin \vartheta}{4 \sinh^2 \rho + 9 \sin^2 \vartheta} (d\chi - 3V)^2. \quad (5.12)$$

The field strength are

$$F_{10}^4 = e^{-4\sigma/3} \mathcal{V}_S^4, \\ F_{10}^3 = e^{-\sigma/3} \frac{1}{k} \sin \vartheta d\vartheta \wedge \mathcal{V}_{S^2}. \quad (5.13)$$

Note that  $F_{10}^4$  is for the D4 branes and  $F_{10}^3$  is for the D6 branes.

The radius of the circle fiber  $y$  is of order

$$e^{4\sigma/3} \sim \frac{N^{1/3}}{k} \sinh \rho. \quad (5.14)$$

As we divide the  $AdS_7$  space, we do not have a small compact circle and so it is hard to say the theory has been reduced to the type IIA theory. However the above radius tells that the M-theory description is valid for  $1 \leq k \lesssim N^{1/3}$ . Since the dilation field diverges at the boundary, the ultraviolet physics at the boundary is the 6d physics. The string frame metric (5.8) in type IIA gives

$$ds_{10}^2 = \frac{R^3}{2k} \left[ (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho ds_{CP^2}^2) + \frac{1}{4} (d\vartheta^2 + \cos^2 \vartheta ds_{S^2}^2) \right]. \quad (5.15)$$

The curvature scale of the type IIA theory is of order  $\sqrt{R^3/2k} \sim \sqrt{N/k}$  which is large when 't Hooft coupling  $\lambda = N/k$  is large.

## 6 Conclusion and Discussion

We have found the supersymmetric Yang-Mills Chern-Simons theories on  $R \times CP^2$  which has the origin from  $Z_k$  modding of the 6d(2,0) theory on  $R \times S^5$  along the circle fiber with a twisting along the  $R$  symmetry direction. Depending on the twisting, the number of supersymmetries can vary from 2,4,6,12. Here we have focused the analysis for 4 supersymmetric case for its simplicity. The fluctuation analysis shows that the fields has the right conformal dimension as expected from the 6d consideration. Supergravity analysis shows that there are M-theory region and type IIA region and weakly coupled region even though the boundary between first two regions is not yet clear.

Our theories are a good ground to calculate the index of the 6d (2,0) theory and we hope to report the result in near future. There seems to be several interesting ideas to pursue from the current point. There may be many BPS objects in the theory which is not apparent in first glance. The  $N^3$  degrees of freedom on the 6d (2,0) theory [28] have been studied from various points of view [14, 17, 29] and our theory may provide a further evidence.

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## A convention for metrics and gamma matrices

The space-time metric has the mostly positive signature. The metric tensors on  $\text{CP}^2$  and  $\text{S}^5$  are, respectively,

$$\begin{aligned} ds_{\text{CP}^2}^2 &= d\rho^2 + \frac{\tau_3^2}{4} \sin^2 \rho \cos^2 \rho + \frac{\tau_1^2 + \tau_2^2}{4} \sin^2 \rho, \\ ds_{\text{S}^5}^2 &= ds_{\text{CP}^2}^2 + (dy + V)^2, \quad V = \frac{\tau_3}{2} \sin^2 \rho, \end{aligned} \quad (\text{A.1})$$

where  $y$  is the  $U(1)$  fiber direction. The left-invariant  $\text{SU}(2)$  1-forms are

$$\begin{aligned} \tau_1 &= -\sin \psi d\theta + \cos \psi \sin \theta d\varphi, \\ \tau_2 &= +\cos \psi d\theta + \sin \psi \sin \theta d\varphi, \\ \tau_3 &= +d\psi + \cos \theta d\varphi, \end{aligned} \quad (\text{A.2})$$

such that  $d\tau_i = \epsilon_{ijk} \tau_j \wedge \tau_k$ . The range of variables are  $\rho \in [0, \frac{\pi}{2}]$ ,  $\theta \in [0, \pi]$ ,  $\varphi \in [0, 2\pi]$ ,  $\psi \in [0, 4\pi]$  and  $y \in [0, 2\pi]$ , the volumes of  $\text{CP}^2$  and  $\text{S}^5$  are  $\pi^2/2$  and  $\pi^3$ , respectively.

The vierbein  $e^m = e_p^m dx^p$  for  $\text{CP}^2$  is

$$e^1 = d\rho, \quad e^2 = \frac{\tau_3}{2} \sin \rho \cos \rho, \quad e^3 = \frac{\tau_1}{2} \sin \rho, \quad e^4 = \frac{\tau_2}{2} \sin \rho. \quad (\text{A.3})$$

Their inverse  $e_m = e_m^p \partial_p$  is

$$e_1 = \partial_\rho, \quad e_2 = \frac{2\tilde{\tau}_3}{\sin \rho \cos \rho}, \quad e_3 = \frac{2\tilde{\tau}_1}{\sin \rho}, \quad e_4 = \frac{2\tilde{\tau}_2}{\sin \rho}, \quad (\text{A.4})$$

where

$$\begin{aligned} \tilde{\tau}_1 &= -\sin \psi \partial_\theta + \frac{\cos \psi}{\sin \theta} (\partial_\varphi - \cos \theta \partial_\psi), \\ \tilde{\tau}_2 &= +\cos \psi \partial_\theta + \frac{\sin \psi}{\sin \theta} (\partial_\varphi - \cos \theta \partial_\psi), \\ \tilde{\tau}_3 &= +\partial_\psi. \end{aligned} \quad (\text{A.5})$$

The Kähler 2-form on  $\mathbb{CP}^2$  is

$$J = \frac{1}{2} J_{mn} e^m \wedge e^n = \frac{1}{2} dV = e^1 \wedge e^2 + e^3 \wedge e^4. \quad (\text{A.6})$$

The spin-connection for the  $\mathbb{CP}^2$  is

$$\begin{aligned} w^{12} &= -\frac{\tau_3}{2} \cos 2\rho, \quad w^{34} = -\frac{\tau_3}{2} (1 + \sin^2 \rho), \\ w^{23} &= w^{41} = +\frac{\tau_2}{2} \cos \rho, \quad w^{31} = w^{42} = +\frac{\tau_1}{2} \cos \rho. \end{aligned} \quad (\text{A.7})$$

The vierbein on  $S^5$  is

$$E^m = e^m \quad (m = 1, 2, 3, 4), \quad E^5 = dy + V_p dx^p. \quad (\text{A.8})$$

The inverse vierbein on  $S^5$  is

$$E_m = e_m - e_m^p V_p \partial_y \quad (m = 1, 2, 3, 4), \quad E_5 = \partial_y. \quad (\text{A.9})$$

The spin connection for  $S^5$  is

$$W^{mn} = w^{mn} - J^{mn} E^5, \quad W^5_m = J_{mn} e^n. \quad (\text{A.10})$$

Our notation for the Minkowski space-time gamma matrices for 6d and 5d is as follows:

$$\begin{aligned} (5d) \quad \gamma^0 &= 1_2 \otimes i\sigma_2, \quad \gamma^{1,2,3} = \sigma_{1,2,3} \otimes \sigma_1, \quad \gamma^4 = 1_2 \otimes \sigma_3, \quad \gamma^{01234} = i1_4, \\ (6d) \quad \Gamma^\mu &= \gamma^\mu \otimes \sigma_1 \quad (\mu = 0, 1, \dots, 4), \quad \Gamma^5 = 1_4 \otimes \sigma_2, \quad \Gamma^7 = \Gamma^{01\dots 5} = -1_4 \otimes \sigma_3. \end{aligned} \quad (\text{A.11})$$

The 6d spinor field  $\lambda$  and the supersymmetric parameter  $\epsilon$  have the opposite chirality so that  $\Gamma^7 \lambda = \lambda$ ,  $\Gamma^7 \epsilon = -\epsilon$ . With  $B = i\sigma_2 \otimes \sigma_1$ , we get  $B\gamma^\mu B^{-1} = -\gamma^{\mu*} = -\gamma_\mu^T$ . The spinors transform as **4** of  $Sp(2)_R = SO(5)$  symmetry and the 5d Euclidean gamma matrices on **4** are

$$\rho_{1,2,3} = \sigma_{1,2,3} \otimes \sigma_3, \quad \rho_4 = 1_2 \otimes \sigma_2, \quad \rho_5 = 1_2 \otimes \sigma_1. \quad (\text{A.12})$$

Our choice of Cartan for  $Sp(2)_R$  is  $R_2 \sim \frac{1}{2}\rho_{45}$  and  $R_1 \sim \frac{1}{2}\rho_{12}$  to fermionic fields. The charge conjugation operator acting on **4** is  $C = i\sigma_2 \otimes \sigma_1$  such that  $C\rho_I C^{-1} = \rho_I^T$ . With  $\hat{B} = B \otimes \sigma_3$ , we get  $\hat{B}\Gamma^M \hat{B}^{-1} = \Gamma^{M*} = \Gamma_M^T$ . We require the the reality conditions on the spinors to be

$$\lambda = -\hat{B}C\lambda^*, \quad \epsilon = \hat{B}C\epsilon^* \implies \lambda = BC\lambda^*, \quad \epsilon = BC\epsilon^* \quad (\text{A.13})$$

on four component spinors.

## B Killing spinors

The Killing spinors [14, 30] on  $R \times S^5$  are defined as follows:

$$\hat{\nabla}_M \epsilon_{\pm} = \frac{i}{2} \Gamma_M \tilde{\epsilon}_{\pm} = \pm \frac{i}{2} \Gamma_M \Gamma_0 \epsilon_{\pm}, \quad (\text{B.1})$$

and  $\epsilon_{\pm} = \hat{B} C \epsilon_{\mp}^*$  and  $\tilde{\epsilon}_{\pm} = \pm \Gamma_0 \epsilon$ . Here we will be loose about 8 and 4 component spinors as the chirality condition  $\Gamma^7 \lambda = \lambda$ ,  $\Gamma^7 \epsilon = -\epsilon$ ,  $\Gamma^7 \tilde{\epsilon} = \tilde{\epsilon}$  leaves no ambiguity. The covariant derivative to the spinor on  $S^5$  given as

$$\nabla_M \epsilon = (\partial_M + \frac{1}{4} W_M^{AB} \Gamma_{AB}) \epsilon. \quad (\text{B.2})$$

The covariant derivative on spinors in  $S^5$  can be expressed in terms of that on  $\text{CP}^2$  plus the derivative along the circle fiber.

$$\begin{aligned} \hat{\nabla}_0 \epsilon &\equiv \partial_t \epsilon = \frac{i}{2} \gamma_0 \tilde{\epsilon} \\ \hat{\nabla}_m \epsilon &\equiv \left[ \nabla_m - V_m \partial_y + \frac{1}{2} J_{mn} \Gamma^{n5} \right] \epsilon = \left[ \nabla_m - V_m \partial_y + \frac{i}{2} J_{mn} \gamma^n \right] \epsilon = \frac{i}{2} \gamma_m \tilde{\epsilon} \\ \hat{\nabla}_5 \epsilon &\equiv \left[ \partial_y - \frac{1}{4} J_{mn} \Gamma^{mn} \right] \epsilon = \left[ \partial_y - \frac{1}{4} J_{mn} \gamma^{mn} \right] \epsilon = \frac{1}{2} \tilde{\epsilon} \end{aligned} \quad (\text{B.3})$$

where  $am = 1, 2, 3, 4$  and  $V = V_m e^m = V_p dx^p$ ,  $J = \frac{1}{2} J_{mn} e^m \wedge e^n$  and  $\nabla_m = e_m^p \nabla_p$  is the covariant derivative on the spinors on  $\text{CP}^2$ . The Killing spinor equation is solved with

$$\tilde{\epsilon} = \pm \Gamma_0 \epsilon = \pm \gamma_0 \epsilon \quad (\text{B.4})$$

The covariant derivative on the gaugino field is

$$\begin{aligned} \hat{\nabla}_0 \lambda &\equiv \partial_t \lambda \\ \hat{\nabla}_m \lambda &\equiv \left[ \nabla_m - V_m \partial_y + \frac{1}{2} J_{mn} \Gamma^{n5} \right] \lambda = \left[ \nabla_m - V_m \partial_y - \frac{i}{2} J_{mn} \gamma^n \right] \lambda \\ \hat{\nabla}_5 \lambda &\equiv \left[ \partial_y - \frac{1}{4} J_{mn} \Gamma^{mn} \right] \lambda = \left[ \partial_y - \frac{1}{4} J_{mn} \gamma^{mn} \right] \lambda \end{aligned} \quad (\text{B.5})$$

Let us split the spinors to eigenspinors  $\gamma_{12} \epsilon^{s_1 s_2} = i s_1 \epsilon^{s_1 s_1}$ ,  $\gamma_{34} \epsilon^{s_1 s_2} = i s_2 \epsilon^{s_1 s_2}$ . Note that  $\gamma^0 \epsilon^{s_1 s_1} = i \gamma^{1234} \epsilon^{s_1 s_2} = -i s_1 s_2 \epsilon^{s_1 s_2}$ . One solution of the Killing spinor is

$$\text{(I)} \quad e_+ \sim e^{-\frac{i}{2}t + \frac{3i}{2}y} \epsilon_0^{++} \quad (\text{B.6})$$

with a constant spinor  $\epsilon_0^{++}$ . It is singlet under the  $SU(3)$  isometry of  $\text{CP}^2$ . The more complicated three Killing spinors are nontrivial linear combinations of three spinors

$$\text{(II)} \quad e_+ \sim e^{-\frac{i}{2}t - \frac{i}{2}y} (e_1^{+-}, \epsilon_1^{-+}, \epsilon_1^{--}) \quad (\text{B.7})$$

where  $\epsilon_1$  depends on  $\text{CP}^2$  coordinates nontrivially. They form a triplet under the  $SU(3)$  isometry of  $\text{CP}^2$ . The detail expression is known but not important here.

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